

## FOLIATIONS, SOLVABILITY AND GLOBAL INJECTIVITY

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ABSTRACT. Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a  $C^\infty$  map such that  $DF(x)$  is invertible for all  $x \in \mathbb{R}^n$ . We know that  $F$  is a local diffeomorphism but, in general, it is not globally injective. We discuss the relations between some additional hypothesis that guarantee the global injectivity of  $F$ . Further, based on one of these hypotheses, we establish a necessary condition for the existence of  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that  $\det DF = h$ , where  $h : \mathbb{R}^n \rightarrow [0, \infty)$  is a given  $C^\infty$  function.

## 1. INTRODUCTION

Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a  $C^\infty$  map such that  $DF(x)$  is invertible for all  $x \in \mathbb{R}^n$ . By the inverse function theorem  $F$  is a local diffeomorphism but, in general, it is not injective. In fact, even in the polynomial case in  $\mathbb{R}^2$ , S. Pinchuk showed in [12] that additional hypotheses are needed for the injectivity of  $F$ . The same problem concerning polynomial maps in  $\mathbb{C}^n$  is yet an open question, widely known as *Jacobian conjecture* (see [1] or [5] for references). In the general case, additional conditions to have the injectivity of  $F$  were established in the literature. The major goal of this paper is to compare some of these additional requirements. Furthermore, based on one of these conditions, we establish a necessary hypothesis for the existence of  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that  $\det DF(x) = h(x)$  for all  $x \in \mathbb{R}^n$ , where  $h : \mathbb{R}^n \rightarrow [0, \infty)$  is a given  $C^\infty$  function. For the statement of our results, we introduce some useful concepts and the injectivity conditions to be related.

In the holomorphic setting, Y. Stein ([13], [14]) proposed a relation between global properties of certain vector fields associated to  $F$  with the Jacobian conjecture for  $n = 2$ . After that, T. Krasinski and S. Spodzieja ([11]) improved this result for the  $n$ -dimensional case. (Early connections has been proposed before, see [1]). More precisely, given a  $C^\infty$  map  $F = (f_1, \dots, f_n) : K^n \rightarrow K^n$ , with  $K = \mathbb{R}$  or  $\mathbb{C}$ , we define  $n$  vector fields  $\mathcal{V}_i$ ,  $i = 1, \dots, n$ , as follows:

$$(1) \quad \mathcal{V}_i(\phi) = \det(D(f_1, \dots, f_{i-1}, \phi, f_{i+1}, \dots, f_n)).$$

Observe that for  $n = 2$ ,  $\mathcal{V}_i = (-1)^i H_{f_j}$ ,  $i \neq j \in \{1, 2\}$ , where  $H_{f_j}$  stands for the Hamiltonian vector field associated to  $f_j$ ,  $H_{f_j} = -\partial_2 f_j \partial_1 + \partial_1 f_j \partial_2$ .<sup>1</sup> We patch together the above mentioned results in the following theorem, where  $E_n$  stands for the set of holomorphic functions  $\mathbb{C}^n \rightarrow \mathbb{C}$ .

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<sup>1</sup>In this paper, for  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , we denote  $\partial_i f = \partial f / \partial x_i$ .

**Theorem 1.1** ([11], [13], [14]). *Let  $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$  be a polynomial map such that  $\det DF$  is a non-zero constant. If  $\mathcal{V}_i(E_n) = E_n$  for  $n - 1$  different indices  $i \in \{1, \dots, n\}$ , then  $F$  is injective.*

In this paper, we consider the vector fields (1) for the  $C^\infty$  real case.

For further reference, we define the notion of *global solvability* of vector fields:

**Definition 1.2.** Let  $M$  be a  $C^\infty$  manifold and  $\mathcal{X} : C^\infty(M) \rightarrow C^\infty(M)$  be a vector field. We say that  $\mathcal{X}$  is *globally solvable* when  $\mathcal{X}(C^\infty(M)) = C^\infty(M)$ .

On the other hand, in Section 2 we prove the following result:

**Proposition 1.3.** *Let  $F = (f_1, \dots, f_n) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a  $C^\infty$  map such that  $\det DF$  is nowhere zero. If there is  $i \in \{1, \dots, n\}$  such that all the intersections*

$$\bigcap_{\substack{j=1 \\ j \neq i}}^n f_j^{-1}(\{c_j\}), \quad c_j \in \mathbb{R},$$

*are connected, then  $F$  is injective.*

These results give rise to the question: When  $\det DF$  is nowhere zero, what are the relations between global solvability of the vector fields  $\mathcal{V}_i$  and the hypothesis of Proposition 1.3? For  $n = 2$ , the answer was given by Theorem 2.4 of [3], where it is shown that for  $i, j \in \{1, 2\}$ ,  $i \neq j$ , the global solvability of  $\mathcal{V}_i$  is equivalent to the connectedness of the level sets of  $f_j$ . We state this result again in Proposition A below. (Actually, in [3] and also as a consequence of our proof of Proposition A below, we have something more precise: If  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is a  $C^\infty$  submersion, then  $H_f$  is globally solvable if and only if  $f^{-1}(\{c\})$  is connected for all  $c \in \mathbb{R}$ ). Our first result, Theorem A, considers such question for  $n \geq 3$ .

For a  $C^\infty$  submersion  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , we denote by  $\mathcal{F}(f)$  the  $C^\infty$  foliation of dimension  $n - 1$  whose leaves are the connected components of the level sets of  $f$ .

**Theorem A.** *Let  $n \geq 3$  and  $F = (f_1, \dots, f_n) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a  $C^\infty$  map such that  $\det DF$  is nowhere zero.*

- (1) *Let  $i \in \{1, \dots, n\}$ . If there are  $i_1, i_2 \in \{1, \dots, n\} \setminus \{i\}$ ,  $i_1 \neq i_2$ , such that for each  $i_k$  ( $k = 1, 2$ ),  $L_{i_k} \in \mathcal{F}(f_{i_k})$  and  $c_j \in \mathbb{R}$ , the set*

$$\left( \bigcap_{j \notin \{i, i_k\}} f_j^{-1}(\{c_j\}) \right) \cap L_{i_k}$$

*is connected, then  $\mathcal{V}_i$  is globally solvable.*

- (2) *Let  $n = 3$  and  $i \in \{1, 2, 3\}$ . If  $\mathcal{V}_i$  is globally solvable, then for  $j, k \in \{1, 2, 3\} \setminus \{i\}$ ,  $j \neq k$ ,  $L_k \in \mathcal{F}(f_k)$  and  $c_j \in \mathbb{R}$ , the set*

$$f_j^{-1}(\{c_j\}) \cap L_k$$

*is connected.*

- (3) *There exist  $J \subset \{1, \dots, n\}$ , with  $\#J = n - 2$ , and  $F$  such that for all  $i \in J$ ,  $\mathcal{V}_i$  is globally solvable but  $\bigcap_{j \neq i} f_j^{-1}(\{c_j\})$  is not connected for some  $c_j \in \mathbb{R}$ .*

**Remark 1.4.** Clearly it follows from statement (1) of the above theorem that if  $\bigcap_{j=1}^n f_j^{-1}(\{c_j\})$  is connected for each  $c_j \in \mathbb{R}$ , then  $\mathcal{V}_i$  is globally solvable.

For  $n = 2$  and  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  the very known concept of *half-Reeb component*, or simply *hRc*, of  $\mathcal{F}(f)$  (we recall it in Definition 3.1) has been used to deal with injectivity. The following result was proved by C. Gutierrez in [8], see also [7] and [9], and was one of the main tools used by him to solve Markus-Yamabe conjecture in [8].

**Proposition 1.5.** *Let  $F = (f_1, f_2) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a  $C^\infty$  map such that  $\det DF$  is nowhere zero. If  $\mathcal{F}(f_1)$  or  $\mathcal{F}(f_2)$  has no hRc, then  $F$  is injective.*

We establish some relations between hRc and global solvability in the following result, which includes the above-mentioned relation between connectedness and solvability.

**Proposition A.** *Let  $F = (f_1, f_2) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a  $C^\infty$  map such that  $\det DF$  is nowhere zero. Then, for  $i, j \in \{1, 2\}$ ,  $i \neq j$ , the following statements are equivalent*

- (1)  $\mathcal{V}_i$  is globally solvable.
- (2)  $\mathcal{F}(f_j)$  does not have any hRc.
- (3)  $f_j^{-1}(\{c\})$  is connected for all  $c \in \mathbb{R}$ .

As a consequence, we obtain a proof of Proposition 1.5 using Proposition 1.3.

Moreover, in [10], C. Gutierrez and C. Maquera defined half-Reeb components of  $\mathcal{F}(f)$  for  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ , as we will recall in Definition 3.4. They used this concept to deal with injectivity (for details, see [10]). Given  $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  such that  $DF(x)$  is invertible for all  $x \in \mathbb{R}^3$ , it is then natural to ask what are the relations between global solvability of  $\mathcal{V}_i$  with the existence or not of half-Reeb components of  $\mathcal{F}(f_j)$  for  $i, j \in \{1, 2, 3\}$ . This is given by our following result.

**Theorem B.** *Let  $F = (f_1, f_2, f_3) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be a  $C^\infty$  map such that  $\det DF$  is nowhere zero. If  $\mathcal{F}(f_i)$  has a hRc, then  $\mathcal{V}_j$  is not globally solvable for  $j \in \{1, 2, 3\} \setminus \{i\}$ .*

The converse of this result fails in general, as we will see in Example 3.7.

Finally, to obtain their main result, the authors of [3] proved the following theorem in a certain special case.

**Theorem 1.6.** *Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a  $C^\infty$  submersion,  $\mathcal{A} \subset \mathbb{R}^2$  be a hRc of  $\mathcal{F}(f)$  and  $U$  a neighborhood of  $\mathcal{A}$ . If  $h : U \rightarrow [0, \infty)$  is a  $C^\infty$  function such that*

$$\int_{\mathcal{A}} h = \infty,$$

*then there exists no  $C^\infty$  function  $g : U \rightarrow \mathbb{R}$  such that  $H_f g = h$  in  $U$ .*

Based on their techniques, we define a new concept that we call *mild half-Reeb component*, or simply *mhRc*, (see Definition 4.1) of a foliation  $\mathcal{F}(f)$ , where  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a  $C^\infty$  submersion. As we will see, in  $\mathbb{R}^2$  and in  $\mathbb{R}^3$ , half-Reeb components are mild half-Reeb components. Our last result is the following generalization of Theorem 1.6.

**Theorem C.** *Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a  $C^\infty$  submersion,  $\mathcal{B} \subset \mathbb{R}^n$  be a mhRc of  $\mathcal{F}(f)$  and  $U$  a neighborhood of  $\mathcal{B}$ . If  $h : U \rightarrow [0, \infty)$  is a  $C^\infty$  function such that*

$$\int_{\mathcal{B}} h = \infty,$$

*then there exists no  $C^\infty$  map  $F = (f_1, \dots, f_n) : U \rightarrow \mathbb{R}^n$  with  $f_1 = f$  and  $\det DF = h$  in  $U$ .*

The paper is organized as follows: In Section 2, we prove some properties of the integral curves of the vector field  $\mathcal{V}_i$ , see Lemma 2.1. As a corollary, we obtain the proof of Proposition 1.3. Then we recall part of a result by J. Duistermaat and L. Hörmander, which gives a geometric characterization of a globally solvable vector field in terms of its integral curves, see Lemma 2.2. With these results we prove statements (1) and (3) of Theorem A.

In Section 3, we recall the definition of hRc in  $\mathbb{R}^2$  and prove Proposition A. Then we state the concept of hRc in  $\mathbb{R}^3$  and prove Theorem B. We also give some examples. We finish the section with the proof of statement (2) of Theorem A.

In Section 4, we recall the main result of [3] and see briefly how it was obtained using Theorem 1.6. Then we define the concept of mhRc and prove Theorem C.

## 2. GLOBAL SOLVABILITY AND CONNECTED COMPONENTS

We begin this section with some properties of integral curves of the vector fields  $\mathcal{V}_i$  defined in (1).

**Lemma 2.1.** *Let  $F = (f_1, \dots, f_n) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a  $C^\infty$  map such that  $\det DF$  is nowhere zero. Then for each  $i \in \{1, \dots, n\}$ ,*

- (1) *The integral curves of  $\mathcal{V}_i$  are the non empty connected components of*
- (2) 
$$\bigcap_{\substack{j=1 \\ j \neq i}}^n f_j^{-1}(\{c_j\}), \quad c_j \in \mathbb{R}.$$

(2) *The function  $f_i$  is strictly monotone along the integral curves of  $\mathcal{V}_i$ .*

(3) *The  $\alpha$ - and  $\omega$ -limit sets in  $\mathbb{R}^n$  of each integral curve of  $\mathcal{V}_i$  are empty.*

*In particular, for each  $i_1 < i_2 < \dots < i_k \in \{1, \dots, n\}$ ,  $k < n$ , all the intersections  $\bigcap_{j=1}^k f_{i_j}^{-1}(\{c_j\})$ ,  $c_j \in \mathbb{R}$ , are empty or unbounded.*

*Proof.* From the definition of  $\mathcal{V}_i$  given in (1), it follows that for each  $j \in \{1, \dots, n\}$ ,

$$\mathcal{V}_i(f_j) = \delta_{ij} \det DF,$$

where  $\delta_{ij}$  stands for the Kronecker delta. Thus giving an integral curve  $\gamma$  of  $\mathcal{V}_i$ , it follows that  $(f_j \circ \gamma)'(t) = \delta_{ij} \det DF(\gamma(t))$ , for all  $t$  in the maximal interval of solution of  $\gamma$ . This shows that  $f_i$  is strictly monotone along  $\gamma$ , proving (2), and that  $\gamma$  is contained in a connected component of an intersection of (2). Since these intersections are 1-dimensional manifolds by the implicit function theorem, the connected component that contains  $\gamma$  must coincide with  $\gamma$ , because  $\mathcal{V}_i$  has no singular points. This proves (1).

To prove (3), we first observe that (1) implies that the integral curves of  $\mathcal{V}_i$  are closed sets. So each integral curve of  $\mathcal{V}_i$  contains its  $\alpha$ - and  $\omega$ -limit sets. If for a integral curve of  $\mathcal{V}_i$  one of these sets was non-empty, we would have a periodic integral curve, a contradiction with (2).  $\square$

*Proof of Proposition 1.3.* Let  $a, b \in \mathbb{R}^n$ ,  $a \neq b$ . If  $f_j(a) = f_j(b) \forall j \neq i$ , then from the hypothesis and statement (1) of Lemma 2.1,  $a$  and  $b$  are in the same integral curve of  $\mathcal{V}_i$ . So, by (2) of Lemma 2.1, it follows that  $f_i(a) \neq f_i(b)$ , which gives  $F(a) \neq F(b)$ .  $\square$

Now we recall part of Theorem 6.4.2 of [6]. It characterizes the global solvability of a given vector field in terms of the geometry of its integral curves.

**Lemma 2.2.** *Let  $M$  be a  $C^\infty$  manifold and  $\mathcal{X} : C^\infty(M) \rightarrow C^\infty(M)$  be a vector field on  $M$ . Then  $\mathcal{X}$  is globally solvable if and only if*

- (1) *No integral curve of  $\mathcal{X}$  is contained in a compact subset of  $M$ , and*
- (2) *For each compact  $K \subset M$ , there exists a compact  $K' \subset M$  such that every compact interval on an integral curve of  $\mathcal{X}$  with end points in  $K$  is contained in  $K'$ .*

*Proof of statement (1) of Theorem A.* Without loss of generality, we assume that  $i = 1$ ,  $i_1 = 2$  and  $i_2 = 3$ . We denote by  $\gamma_x$  the integral curve of  $\mathcal{V}_1$  through  $x \in \mathbb{R}^n$ . Suppose on the contrary that  $\mathcal{V}_1$  is not globally solvable. From (3) of Lemma 2.1, no integral curve of  $\mathcal{V}_1$  is contained in a compact subset of  $\mathbb{R}^n$ . Thus from Lemma 2.2, there exist sequences  $\{x_k\} \subset \mathbb{R}^n$  and  $\{t_k\}, \{s_k\} \subset \mathbb{R}$ , with  $0 < s_k < t_k$ , and  $a, b \in \mathbb{R}^n$  such that

$$(2) \quad x_k \rightarrow a, \quad \gamma_{x_k}(t_k) \rightarrow b, \quad |\gamma_{x_k}(s_k)| > k.$$

We take  $\gamma_a$  and  $\gamma_b$  the integral curves of  $\mathcal{V}_1$  through  $a$  and  $b$  respectively. Using the facts that the level sets of  $f_1$  are local transversals to the flow of  $\mathcal{V}_1$  and  $f_1$  is monotone along each integral curve of  $\mathcal{V}_1$  (from (2) of Lemma 2.1), we conclude from (3) and from the flow box theorem that  $\gamma_a$  and  $\gamma_b$  are different integral curves.

Now considering  $c_2, \dots, c_n \in \mathbb{R}$  such that  $\gamma_a$  and  $\gamma_b$  are distinct connected components of  $\cap_{j=2}^n f_j^{-1}(\{c_j\})$ , it follows from the hypothesis that  $\gamma_a$  and  $\gamma_b$  are in distinct connected components of  $f_3^{-1}(\{c_3\})$ . Hence  $\gamma_a$  and  $\gamma_b$  are in two distinct connected components of  $\cap_{j=3}^n f_j^{-1}(\{c_j\})$ , which we denote by  $\Gamma_a$  and  $\Gamma_b$ , respectively. Then we take open neighbourhoods  $N_a$  and  $N_b$  of  $a$  and  $b$  respectively such that all the leaves of the foliation  $\mathcal{F}(f_2)$  in  $N_a$  intersect  $\Gamma_a$  and all the leaves of  $\mathcal{F}(f_2)$  in  $N_b$  intersect  $\Gamma_b$  (this is possible since  $\mathcal{F}(f_2)$  is transversal to the foliation given by the connected components of  $\cap_{j=3}^n f_j^{-1}(\{c_j\})$ ). We then take  $k$  big enough in order that  $x_k \in N_a$  and  $\gamma_{x_k}(t_k) \in N_b$  and take  $L_2 \in \mathcal{F}(f_2)$  containing  $\gamma_{x_k}$ . This leaf  $L_2$  intersects  $\Gamma_a$  and  $\Gamma_b$  and thus  $\cap_{j=3}^n f_j^{-1}(\{c_j\}) \cap L_2$  is disconnected, a contradiction with the hypothesis.  $\square$

The proof of statement (2) of Theorem A will be given at the end of Section 3.

*Proof of statement (3) of Theorem A.* Let

$$F(x) = (e^{x_1} \cos x_2, e^{x_1} \sin x_2, x_3, \dots, x_n).$$

We have  $\det DF(x) = e^{2x_1} > 0$  for all  $x \in \mathbb{R}^n$ . Moreover, for each  $i = 3, \dots, n$ , it is clear that  $\mathcal{V}_i = e^{2x_1} \partial_i$  is a globally solvable vector field. On the other hand, taking  $c_1 = 1$ ,  $c_2 = 0$  and  $c_3, \dots, c_n \in \mathbb{R}$ , it follows that for each  $i = 3, \dots, n$ , the set

$$\bigcap_{j \neq i} f_j^{-1}(\{c_j\}) = \bigcup_{k \in \mathbb{Z}} \{(0, 2k\pi, c_3, \dots, c_{i-1}, z, c_{i+1}, \dots, c_n) \mid z \in \mathbb{R}\}$$

is disconnected.  $\square$

### 3. HALF-REEB COMPONENTS IN $\mathbb{R}^2$ AND $\mathbb{R}^3$ . RELATIONS WITH GLOBAL SOLVABILITY

Given a  $C^\infty$  submersion  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ , we recall now the concept of half-Reeb component of the foliation  $\mathcal{F}(f)$ .

**Definition 3.1.** Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a  $C^\infty$  submersion,  $h_0(x, y) = xy$  and  $B = \{(x, y) \in [0, 2] \times [0, 2] \mid 0 < x + y \leq 2\}$ . We say that  $\mathcal{A} \subset \mathbb{R}^2$  is a *half-Reeb component*, or simply a *hRc*, of  $\mathcal{F}(f)$  if there is a homeomorphism  $T : B \rightarrow \mathcal{A}$  which is a topological equivalence between  $\mathcal{F}(h_0)|_B$  and  $\mathcal{F}(f)|_{\mathcal{A}}$  with the following properties:

- (1) The segment  $\{(x, y) \in B \mid x + y = 2\}$  is sent by  $T$  onto a transversal section for the leaves of  $\mathcal{F}(f)$  in the complement of  $T(1, 1)$ . This section is called the *compact edge* of  $\mathcal{A}$ .
- (2) Both segments  $\{(x, y) \in B \mid x = 0\}$  and  $\{(x, y) \in B \mid y = 0\}$  are sent by  $T$  onto full half-leaves of  $\mathcal{F}(f)$ . These two half-leaves are called the *non-compact edges* of  $\mathcal{A}$ .

It is a known fact that the leaves of  $\mathcal{F}(f)$  are the integral curves of  $H_f$  (in Lemma 2.1 this is proved when  $f$  is a component of a local invertible map  $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ ). Before the proof of Proposition A, we describe the half-Reeb component  $\mathcal{A}$  by using the integral curves of  $H_f$ . Given  $x \in \mathbb{R}^2$ , we denote by  $\gamma_x(t)$  the integral curve of  $H_f$  such that  $\gamma_x(0) = x$ . We denote by  $S$  an arc with end-points  $p$  and  $q$  in two different integral curves of the vector field  $H_f$  such that there exists a point  $w \in S \setminus \{p, q\}$  with the following properties:

- (1) The arcs  $S_1 \subset S$ , from  $p$  to  $w$ , and  $S_2 \subset S$ , from  $q$  to  $w$ , are transversal sections to the flow of  $H_f$  away from the point  $w$ .
- (2) For every point  $x$  in  $S_1 \setminus \{p, w\}$  the integral curve  $\gamma_x$  crosses a point  $y(x)$  in  $S_2 \setminus \{q, w\}$ .
- (3) The map  $S_1 \setminus \{p, w\} \ni x \mapsto y(x) \in S_2 \setminus \{q, w\}$  is a homeomorphism and extend homeomorphically to  $S_1$  by setting  $y(p) = q$  and  $y(w) = w$ .

We define  $G$  to be the union of the intervals from  $x$  to  $y(x)$  of  $\gamma_x$ , for all  $x \in S_1 \setminus \{p\}$ , and take  $\mathcal{A} = G \cup (\gamma_p \cap \overline{G}) \cup (\gamma_q \cap \overline{G})$ . It is clear that this set is a hRc of  $\mathcal{F}(f)$  with compact edge being the arc  $S$  and non-compact edge the half-solutions  $\gamma_p \cap \overline{G}$  and  $\gamma_q \cap \overline{G}$ . It is also clear that a hRc of  $\mathcal{F}(f)$  satisfies the properties of the set just constructed.

We observe also that the existence of hRc is equivalent to the existence of inseparable leaves on the foliation  $\mathcal{F}(f)$ , see [9] for details.

*Proof of Proposition A.* In this proof, given  $x \in \mathbb{R}^2$ , we denote by  $I_x$  the maximal interval of solution of  $\gamma_x$ , where as above  $\gamma_x(t)$  is the integral curve of  $H_{f_j} = (-1)^i \mathcal{V}_i$  such that  $\gamma_x(0) = x$ . Further, we denote  $\gamma_x^+ = \{\gamma_x(t) \mid t \geq 0, t \in I_x\}$  and  $\gamma_x^- = \{\gamma_x(t) \mid t \leq 0, t \in I_x\}$ . Finally, given a set  $A \subset \mathbb{R}^2$ , we say that the *positive end* (respectively *negative end*) of  $\gamma_x$  is in  $A$  when there exists  $t_0 \in I_x$ ,  $t_0 \geq 0$  (respectively  $t_0 \leq 0$ ) such that  $\gamma_x(t) \in A$  for all  $t > t_0$ ,  $t \in I_x$ , (respectively for all  $t < t_0$ ,  $t \in I_x$ ).

It is clear that (1) implies (2), since appearance of a hRc in the foliation  $\mathcal{F}(f_j)$ , according to Lemma 2.1, would make condition (2) in Lemma 2.2 to fail.

The fact that (3) implies (1) follows as in the proof of (1) of Theorem A: if  $\mathcal{V}_i$  is not globally solvable, it follows from Lemma 2.2 that there exist sequences  $\{x_k\} \subset \mathbb{R}^2$  and  $\{t_k\}, \{s_k\} \subset \mathbb{R}$  such that  $x_k \rightarrow a$ ,  $\gamma_{x_k}(t_k) \rightarrow b$  and  $|\gamma_{x_k}(s_k)| > k$ . Then the curves  $\gamma_a$  and  $\gamma_b$  are different (use the Flow Box Theorem), but they belong to the same level set of  $f_j$ , a contradiction.

Thus it remains to prove that (2) implies (3). We assume (2) and suppose on the contrary that there exists  $c \in \mathbb{R}$  such that  $f_j^{-1}(\{c\})$  is not connected. We will

construct a hRc of  $\mathcal{F}(f_j)$  prescribing a set  $\mathcal{A}$  as explained right after Definition 3.1, obtaining a contradiction with (2). We take  $p$  and  $q$  in two distinct connected components of  $f_j^{-1}(\{c\})$  and denote by  $\Gamma$  the open connected region of  $\mathbb{R}^2$  bounded by the integral curves  $\gamma_p$  and  $\gamma_q$  (recall (1) of Lemma (2.1)). Let  $\lambda : [0, 1] \rightarrow \mathbb{R}^2$  be a  $C^\infty$  injective curve such that  $\lambda(0) = p$ ,  $\lambda(1) = q$  and  $\lambda((0, 1)) \subset \Gamma$ . The curve  $\lambda$  separates  $\Gamma$  in two open connected regions that we denote by  $\Gamma_1$  and  $\Gamma_2$ , respectively.

Either the global maximum or the global minimum of  $f_j \circ \lambda(t)$  is attained at a point  $t_m \in (0, 1)$ . Hence  $\gamma_{\lambda(t_m)}$  is entirely contained in  $\Gamma_1 \cup \lambda([0, 1])$  or in  $\Gamma_2 \cup \lambda([0, 1])$ . In particular, both the positive and the negative ends of  $\gamma_{\lambda(t_m)}$  are contained in  $\Gamma_1$  or in  $\Gamma_2$ , respectively. Therefore, the set

$$T = \{t \in (0, 1) \mid \text{both the ends of } \gamma_{\lambda(t)} \text{ are contained in } \Gamma_1 \text{ or in } \Gamma_2\}$$

is non-empty. Let  $t_1 \in [0, 1]$  be the greatest lower bound of  $T$ . We have two possibilities: either  $t_1 \in T$  or  $t_1 \notin T$ .

In the first possibility, we suppose without loss of generality that both the ends of  $\gamma_{\lambda(t_1)}$  are in  $\Gamma_1$ . It is clear from the definition of  $t_1$  that  $t_1 > 0$  and that  $\gamma_{\lambda(t_1)}$  does not intersect  $\lambda([0, t_1])$ . We take a cross section to the flow of  $H_{f_j}$ ,  $\alpha : [a, b] \rightarrow \mathbb{R}^2$ , such that  $\alpha(a) = \lambda(t_1)$  and such that  $\alpha((a, b])$  and  $\lambda((0, t_1))$  are contained in the same connected region of  $\Gamma$  defined by  $\gamma_{\lambda(t_1)}$ . For  $b' \in (a, b)$  close enough to  $a$ , we conclude that the integral curves of  $H_{f_j}$  crossing  $\alpha((a, b'))$  have one end in  $\Gamma_1$  and the other end in  $\Gamma_2$ . Without loss of generality, we assume that the negative ends of these curves are in  $\Gamma_1$ . From continuous dependence, by taking  $b'$  smaller if necessary, we can assume that for all  $r \in (a, b')$ ,

$$s_r = \sup\{s \mid \gamma_{\alpha(r)}(s) \in \Gamma_1 \cup \lambda\} \in (0, \infty).$$

Clearly there is  $t(r) \in (0, 1)$  such that

$$(4) \quad \gamma_{\alpha(r)}(s_r) = \lambda(t(r)), \quad \gamma_{\alpha(r)}(s) \in \Gamma_2, \quad \forall s > s_r, \quad s \in I_{\alpha(r)}.$$

The function  $(a, b') \ni r \mapsto t(r)$  is clearly injective. We claim that this function is also monotone. The claim follows if we prove that: for each  $r_1, r_2, r_3 \in (a, b')$  with  $r_1 < r_2 < r_3$ , the point  $t(r_2)$  is contained in the interval determined by  $t(r_1)$  and  $t(r_3)$ . To prove this, consider the open set  $A$  contained in  $\Gamma_2$  bounded by the interval of curves  $\gamma_{\lambda(t(r_1))}^+$ ,  $\gamma_{\lambda(t(r_3))}^+$  and the interval of  $\lambda(t)$  with end points  $\lambda(t(r_1))$  and  $\lambda(t(r_3))$ . The curve  $\gamma_{\alpha(r_2)}$  is contained in the region bounded by  $\gamma_{\alpha(r_1)}$  and  $\gamma_{\alpha(r_3)}$ . In particular, this curve will enter the set  $A$  according to (4). The only way to do that is crossing the interval of  $\lambda(t)$  with end points  $\lambda(t(r_1))$  and  $\lambda(t(r_3))$ . This proves that  $t(r_2)$  is between  $t(r_1)$  and  $t(r_3)$ .

From the claim it follows that there exists  $t_2 = \lim_{r \rightarrow a^+} t(r)$ . We consider the integral curve  $\gamma_{\lambda(t_2)}$ . We take a cross section  $\beta : [c, d] \rightarrow \mathbb{R}^2$  with  $\beta(c) = \lambda(t_2)$  and  $\beta((c, d])$  contained in the open connected region defined by  $\gamma_{\lambda(t_2)}$  containing the curves  $\gamma_{\alpha(r)}$ . For  $r_0$  close enough to  $a$ , we conclude by construction that all the curves  $\gamma_{\alpha(r)}$ ,  $r \in (a, r_0]$ , intersect  $\beta((c, d])$ . Let  $u_0 \in (c, d]$  and  $s_0 > 0$  such that  $\gamma_{\alpha(r_0)}(s_0) = \beta(u_0)$ . We take the curve

$$S = \alpha([a, r_0]) \cup \gamma_{\alpha(r_0)}([0, s_0]) \cup \beta([c, u_0]).$$

Using the Flow Box Theorem along the interval of curve  $\gamma_{\alpha(r_0)}([0, s_0])$ , we can modify the continuous curve  $S$  to a  $C^\infty$  curve  $S'$  which is transversal to the flow of  $H_{f_j}$  up to a point  $w'$ . From construction, the arc  $S'$ , with end-points  $p' = \lambda(t_1)$

and  $q' = \lambda(t_2)$ , satisfy the properties (1), (2) and (3) prescribed after Definition 3.1. Thus we obtain a hRc of  $\mathcal{F}(f_j)$  with compact edge being  $S'$  and non-compact edges being  $\gamma_{\lambda(t_1)}^+$  and  $\gamma_{\lambda(t_2)}^-$  finishing the proof in case  $t_1 \in T$ .

Now in the case  $t_1 \notin T$ , it follows that  $\gamma_{\lambda(t_1)}$  has one end in  $\Gamma_1$  and one end in  $\Gamma_2$  or  $t_1 = 0$ . In both cases, we take a cross section  $\alpha : [a, b] \rightarrow \mathbb{R}^2$  such that  $\alpha(a) = \lambda(t_1)$  and  $\alpha((a, b])$  is contained in the open region determined by  $\gamma_{\lambda(t_1)}$  containing  $q$ . From the definition of  $t_1$  there exists  $b' \in (a, b)$  such that each integral curve of  $H_{f_j}$  crossing  $\alpha((a, b'))$  has its both ends in  $\Gamma_1$  or in  $\Gamma_2$ . We assume without loss of generality that they have its both ends in  $\Gamma_1$ . Assuming that the negative end of  $\gamma_{\lambda(t_1)}$  is in  $\Gamma_1$ , it follows from continuous dependence, taking  $b'$  smaller if necessary, that for all  $r \in (a, b')$ ,

$$s_r = \sup\{s \mid \gamma_{\alpha(r)}(s) \in \Gamma_2 \cup \lambda\} \in (0, \infty).$$

As above we define  $t(r)$  to be the point of  $(0, 1)$  such that  $\gamma_{\alpha(r)}(s_r) = \lambda(t(r))$ . Then similar arguments show that  $t(r)$  is monotone and as above we construct a hRc with one of its non-compact edges being  $\gamma_{\lambda(t_1)}^+$ .  $\square$

**Remark 3.2.** In the proof of Proposition A above we did not use the full hypothesis that  $f_j$  is one of the components of a local invertible map  $(f_1, f_2) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ . We only used that  $f_j$  is a submersion (and as a consequence the leaves of  $\mathcal{F}(f_j)$  are the integral curves of the vector field  $H_{f_j}$ ). Hence we have proved that *a  $C^\infty$  submersion  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  has a disconnected level set if and only if  $\mathcal{F}(f)$  has a hRc if and only if  $H_f$  is not globally solvable.*

Moreover, if  $f^{-1}(\{c\})$  is a disconnected level set of the submersion  $f$ , let as in the above proof two connected components  $\gamma_p$  and  $\gamma_q$  of  $f^{-1}(\{c\})$ . Let also  $\Gamma$  be the open connected set whose border is  $\gamma_p \cup \gamma_q$ , and  $\lambda$  be a curve contained in  $\bar{\Gamma}$  connecting  $\gamma_p$  and  $\gamma_q$ . Our proof above shows that under these hypotheses, *there exists a hRc of  $\mathcal{F}(f)$  contained in  $\bar{\Gamma}$  whose non-compact edges intercept the curve  $\lambda$ .*

Now we recall the concept of half-Reeb component for a  $C^\infty$  submersion  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  as introduced in [10]. We first define the concept of *vanishing cycle* for  $\mathcal{F}(f)$ .

**Definition 3.3.** Given a  $C^\infty$  submersion  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ , we say that a  $C^\infty$  embedding  $F_0 : S^1 \rightarrow \mathbb{R}^3$  is a *vanishing cycle* for  $\mathcal{F}(f)$  if it satisfies:

- (1)  $F_0(S^1)$  is contained in a leaf  $L_0$  of  $\mathcal{F}(f)$  and is *not* homotopic to a point in  $L_0$ .
- (2)  $F_0$  can be extended to a  $C^\infty$  embedding  $F : [-1, 2] \times S^1 \rightarrow \mathbb{R}^3$  such that for all  $t \in (0, 1]$  there is a 2-disc  $D_t$  contained in a leaf  $L_t$  with  $\partial D_t = F(\{t\} \times S^1)$ .
- (3) For all  $x \in S^1$  the curve  $t \in [-1, 2] \mapsto F(t, x)$  is transverse to the foliation  $\mathcal{F}(f)$ , and for all  $t \in (0, 1]$  the disc  $D_t$  depends continuously on  $t$ .

We say that the leaf  $L_0$  *supports* the vanishing cycle  $F_0$  and that the map  $F$  is *associated* to  $F_0$ .

**Definition 3.4.** The *half-Reeb component*, or simply *hRc*, of  $\mathcal{F}(f)$  associated to the vanishing cycle  $F_0$  is the region

$$\mathcal{A} = (\cup_{t \in (0, 1]} D_t) \cup L \cup F_0(S^1),$$



where  $L$  is the connected component of  $L_0 \setminus F_0(S^1)$  contained in the closure of  $\cup_{t \in (0,1]} D_t$ .

We say that the transversal  $F([0,1] \times S^1)$  (to the foliation  $\mathcal{F}(f)$ ) is the *compact face* of  $\mathcal{A}$  and  $L \cup F_0(S^1)$  is the *non-compact face* of  $\mathcal{A}$ .

**Example 3.5.** Let  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$  be defined by  $g(x, y) = x(1 - xy^2)$ . It is easy to see that  $g$  is a submersion and that the set  $\{(x, y) \mid -1/\sqrt{x} \leq y \leq 1/\sqrt{x}, 0 < x \leq 1\}$  is a hRc of  $\mathcal{F}(g)$  with compact edge being the segment  $\{(1, y) \mid y \in [-1, 1]\}$  and non-compact edges being the curves  $y = 1/\sqrt{x}$  and  $y = -1/\sqrt{x}$ , with  $x \in (0, 1)$ .

Now we rotate this function around the  $x$ -axis and obtain a hRc in  $\mathbb{R}^3$ . More precisely, define  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  by  $f(x, y, z) = g(x, y^2 + z^2) = x - x^2(y^2 + z^2)^2$ . It is easy to see that  $f$  is a submersion and that the set  $\{(x, y, z) \mid y^2 + z^2 \leq 1/\sqrt{x}, 0 < x \leq 1\}$  is a hRc of  $\mathcal{F}(f)$ , now according to Definition 3.4. Here the compact face is the set  $\{(1, y, z) \mid y^2 + z^2 \leq 1\}$  and the non-compact face is the set  $\{(x, y, z) \mid y^2 + z^2 = 1/\sqrt{x}, 0 < x \leq 1\}$ .

**Lemma 3.6.** *Let  $\mathcal{A}$  be a hRc in the sense of Definition 3.4. Then  $\text{int}(\mathcal{A})$  is unbounded.*

*Proof.* Suppose on the contrary that  $\text{int}(\mathcal{A})$  is bounded. So  $\overline{L}$  is compact, and hence we can cover it by a tubular neighborhood  $U$  and construct  $\pi : U \rightarrow \overline{L}$  a  $C^\infty$  submersion with the property that  $\pi(w) = w$  and  $\pi^{-1}(\{w\})$  is transversal to  $\mathcal{F}(f)$  for each  $w \in \overline{L}$  (see Lemma 2 of Chapter 4 of [4]). It is clear that this submersion can be taken so that  $\pi^{-1}(F_0(S^1)) \cap D_t = \partial D_t$  for each  $t \in (0, \delta)$ , for some  $\delta > 0$ .

Using continuity, we can choose  $t \in (0, \delta)$  small enough so that  $D_t \subset U$ . We fix  $p_0 \in L$  and define  $p_t$  to be the unique point of intersection of  $\pi^{-1}(\{p_0\})$  with  $D_t$ . Since  $D_t$  is contractible, there exists a  $C^\infty$  map  $H_t : S^1 \times [0, 1] \rightarrow D_t$  such that  $H_t(x, 0) = F(t, x)$  and  $H_t(x, 1) = p_t$  for all  $x \in S^1$ . We define  $H : S^1 \times [0, 1] \rightarrow \overline{L}$  by  $H(x, s) = \pi \circ H_t(x, s)$ . This is a  $C^\infty$  map such that  $H(x, 0) = F(0, x)$  and  $H(x, 1) = p_0$  for all  $x \in S^1$ , which proves that  $F_0(S^1)$  is homotopic to  $p_0$ , a contradiction with (1) of Definition 3.3.  $\square$

*Proof of Theorem B.* Consider  $\mathcal{A}$  to be a hRc of  $\mathcal{F}(f_i)$ . We will use the notation of definitions 3.3 and 3.4. Let  $K = F([0, 2] \times S^1)$ , where  $F_0 : S^1 \rightarrow \mathbb{R}^3$  is a vanishing cycle for the foliation  $\mathcal{F}(f_i)$ . For each  $n \in \mathbb{N}$ , let  $x_n \in \text{int}(\mathcal{A})$  such that  $x_n \notin B(0, n)$  (this is possible from Lemma 3.6). For  $n$  sufficiently large, there is  $t_n \in (0, 2]$  such that  $x_n \in D_{t_n}$ . Let  $j \in \{1, 2, 3\}$ ,  $j \neq i$ , and consider  $\gamma_{x_n}$  the integral curve of  $\mathcal{V}_j$  passing through  $x_n$ . From (1) of Lemma 2.1,  $\gamma_{x_n} \subset L_{t_n}$ . Since  $D_{t_n}$  is bounded, the statement (3) of the same lemma asserts the existence of  $s_n^1, s_n^2$ ,  $s_n^1 < 0 < s_n^2$ , in the interval of definition of  $\gamma_{x_n}$ , such that

$$\gamma_{x_n}(s_n^\tau) \in \partial D_{t_n} \subset K, \text{ for } \tau = 1, 2.$$

This shows that  $\mathcal{V}_j$  is not globally solvable by Lemma 2.2.  $\square$

On the other hand, the converse of Theorem B is not true in general, as we can see in the following example.

**Example 3.7.** We take again the map  $F(x) = (e^{x_1} \cos x_2, e^{x_1} \sin x_2, x_3)$ . For  $i = 1, 2, 3$ , each leaf of the foliation  $\mathcal{F}(f_i)$  is homeomorphic to  $\mathbb{R}^2$ , and hence it follows that  $\mathcal{F}(f_i)$  has no half-Reeb components. But  $\mathcal{V}_1$  and  $\mathcal{V}_2$  are not globally solvable. To see this we observe that for all  $c > 0$ , the sets  $\{(x_1, x_2, 0) \mid e^{x_1} \sin x_2 = c, x_2 \in (0, \pi)\}$  are integral curves of  $\mathcal{V}_1$ , according to (1) of Lemma 2.1. Taking

$K = \{0\} \times [0, \pi] \times \{0\}$ , we observe there is no compact  $K'$  as in condition (2) of Lemma 2.2. We can use the same idea to prove that  $\mathcal{V}_2$  is not globally solvable.

**Remark 3.8.** Since the map  $F$  in the last example is not injective, we see that the non-existence of hRc on the foliations  $\mathcal{F}(f_i)$  does not guarantee global injectivity.

**Remark 3.9.** When  $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is a diffeomorphism,  $\mathcal{F}(f_i)$  does not have any hRc for  $i = 1, 2, 3$ , because all its leaves are diffeomorphic to  $\mathbb{R}^2$ . So in the polynomial case, where injectivity implies surjectivity, by [2], for example, the existence of hRc is an obstruction for injectivity.

On the other hand, for  $n = 3$ , the existence of hRc does not interfere on the injectivity of  $F$  in general:

**Example 3.10.** Let  $f_1 : \mathbb{R}^3 \rightarrow \mathbb{R}$  be given by  $f_1(x) = x_1^2 + x_2^2 - e^{x_3}$ . We have that  $f_1$  is a submersion and that the set  $\{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 \leq e^{x_3}, x_3 \leq 0\}$  is a half-Reeb component of  $\mathcal{F}(f_1)$  with compact face  $\{(x_1, x_2, 0) \mid x_1^2 + x_2^2 \leq 1\}$  and  $F_0(S^1) = \{(x_1, x_2, 0) \mid x_1^2 + x_2^2 = 1\}$  (contained in the leaf  $x_1^2 + x_2^2 = e^{x_3}$ ). Let  $G = (f_2, f_3) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a  $C^\infty$  map such that  $\det DG$  is nowhere zero and define  $F(x) = (f_1(x), G(x_1, x_2))$ . We can have two situations:

- (1)  $F$  is injective if  $G$  is injective.
- (2)  $F$  is not injective if  $G$  is not injective.

Take  $G = (x_1, x_2)$  for item (1) and  $G = (e^{x_1} \cos x_2, e^{x_1} \sin x_2)$ , for item (2), for instance.

*Proof of statement (2) of Theorem A.* Without loss of generality we assume that  $i = 1, k = 2$  and  $j = 3$ . From Theorem B it follows that  $\mathcal{F}(f_2)$  does not have hRc. From Proposition 2.2 of [10] it follows that all the leaves of  $\mathcal{F}(f_2)$  are diffeomorphic to  $\mathbb{R}^2$ .

We suppose on the contrary that there are  $c_3 \in \mathbb{R}$  and  $L_2 \in \mathcal{F}(f_2)$  such that  $f_3^{-1}(\{c_3\}) \cap L_2$  is not connected. We consider a diffeomorphism  $H : L_2 \rightarrow \mathbb{R}^2$  and the map  $G = (g_1, g_2) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined by  $G(x) = (f_1, f_3) \circ H^{-1}(x)$ . It follows that  $\det DG(x) \neq 0$  in  $\mathbb{R}^2$ , because  $f_1$  and  $f_3$  are transversal to  $L_2$ , and that  $g_2^{-1}(\{c_3\})$  is not connected. From Proposition A, the foliation  $\mathcal{F}(g_2)$  of  $\mathbb{R}^2$  has a hRc. Thus going back to  $L_2$ , we have, in the light of Lemma 2.1, integral curves of  $\mathcal{V}_1$  violating item (2) of Lemma 2.2, a contradiction with the global solvability of  $\mathcal{V}_1$ .  $\square$

#### 4. A NECESSARY CONDITION FOR LOCAL INVERTIBILITY

As we have said in the introduction section, a version of Theorem 1.6 was used to obtain the main result of [3] which says: If  $F = (f, g)$  is a polynomial map such that  $\det DF(x) > 0, \forall x \in \mathbb{R}^2$ , and  $\deg f \leq 3$ , then  $F$  is injective. We begin this section explaining briefly how this result was proved in [3]: The authors supposed by contradiction that  $H_f$  is not globally solvable (recall that if it is globally solvable, the injectivity of  $F$  follows from propositions 1.5 and A). Then they proved that, up to an affine change of coordinates,  $f(x, y) = x(1 + xy)$ . In the sequel, they proved that  $\int_{\mathcal{A}} h = \infty$  for all strictly positive polynomial  $h$ , where  $\mathcal{A}$  is a hRc of  $\mathcal{F}(f)$ , for this special  $f$  (the existence of  $\mathcal{A}$  is given by Proposition A). Since  $\det DF = H_f g$ , they got a contradiction with their version of Theorem 1.6, which is this theorem for the special case of a hRc of  $\mathcal{F}(x(1 + xy))$ .

We now define the concept of mild half-Reeb component in order to state and prove Theorem C.

**Definition 4.1.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a  $C^\infty$  submersion. We say that a region  $\mathcal{B} \subset \mathbb{R}^n$  is a *mild half-Reeb component*, or simply a *mhRc*, of the foliation  $\mathcal{F}(f)$  if it satisfies

$$\mathcal{B} = \overline{\bigcup_{k \in \mathbb{N}} P_k},$$

where  $P_k$  is a bounded connected open set whose boundary is  $C^1$  by parts and given by the union of a subset  $Q_k$  of a leaf of  $\mathcal{F}(f)$  and a hypersurface  $L_k$  of  $\mathbb{R}^n$ . Furthermore,  $P_k \subset P_{k+1}$  for all  $k \in \mathbb{N}$  and there exists a *bounded* hypersurface  $L$  of  $\mathbb{R}^n$  such that  $L_k \subset L_{k+1} \subset L$  for all  $k \in \mathbb{N}$ .

**Example 4.2.** Half-Reeb components in  $\mathbb{R}^2$  and  $\mathbb{R}^3$  are mild half-Reeb components.

To prove Theorem C, we will use the vector fields  $\mathcal{V}_i$  calculated in coordinates:

$$(5) \quad \mathcal{V}_i = \sum_{j=1}^n \text{coff}(a_{ij}) \partial_j,$$

where  $\text{coff}(a_{ij})$  is the cofactor of the entry  $a_{ij}$  in the matrix  $DF$ . We need the following technical result.

**Lemma 4.3.** Let  $F = (f_1, \dots, f_n) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a  $C^\infty$  map. Then

$$\text{div}(f_i \mathcal{V}_i) = \det DF,$$

for all  $i = 1, \dots, n$ .

*Proof.* If we denote  $\mathcal{V}_{F,i} = \mathcal{V}_i$  to emphasize the dependence of the map  $F$ , it is easy to see that  $\mathcal{V}_{F,i} = -\mathcal{V}_{\overline{F},n}$ , where  $\overline{F}$  is the map  $F$  with  $f_i$  permuted with  $f_n$ . So, since  $\det(DF) = -\det(D\overline{F})$ , it is enough to prove that  $\text{div}(f_n \mathcal{V}_n) = \det DF$ . By (5) we have

$$\text{div}(f_n \mathcal{V}_n) = \sum_{k=1}^n (\partial_k f_n) \text{coff}(a_{nk}) + f_n \sum_{k=1}^n \partial_k \text{coff}(a_{nk}).$$

It is clear that the first term above is  $\det DF$ . We assert that the second term is zero. Indeed, we have

$$\sum_{k=1}^n \partial_k \text{coff}(a_{nk}) = \sum_{k=1}^n \sum_{\substack{j=1 \\ j \neq k}}^n S_{kj},$$

where

$$S_{kj} = (-1)^{n+k} \det \left( \partial_1 g, \dots, \partial_{j-1} g, \partial_k \partial_j g, \partial_{j+1} g, \dots, \widehat{\partial_k g}, \dots, \partial_n g \right)$$

and  $g = (f_1, \dots, f_{n-1})^t$ , see (5). So it is enough to prove that  $S_{kj} = -S_{jk}$ .

Suppose first that  $k < j$ . We have

$$\begin{aligned} S_{kj} &= (-1)^{n+k} \det (\partial_1 g, \dots, \partial_{k-1} g, \partial_{k+1} g, \dots, \partial_{j-1} g, \partial_k \partial_j g, \partial_{j+1} g, \dots, \partial_n g) \\ &= (-1)^{n+k} (-1)^{j-1-k} \det (\dots, \partial_{k-1} g, \partial_k \partial_j g, \partial_{k+1} g, \dots, \partial_{j-1} g, \partial_{j+1} g, \dots) \\ &= -S_{jk}. \end{aligned}$$

The case  $k > j$  is similar. □

*Proof of Theorem C.* Suppose on the contrary that there are  $f_2, \dots, f_n$  with the properties required. We claim that there exists  $M \in \mathbb{R}$  such that

$$\int_{P_k} h \leq M, \text{ for all } k \in \mathbb{N}.$$

This will be a contradiction with the hypothesis by the monotone convergence theorem. Therefore it remains to prove the claim. By the divergence theorem

$$\int_{P_k} \operatorname{div}(f_n \mathcal{V}_n) = \int_{Q_k} \langle f_n \mathcal{V}_n, N \rangle w_{Q_k} + \int_{L_k} \langle f_n \mathcal{V}_n, N \rangle w_{L_k},$$

where  $w_{Q_k}$  and  $w_{L_k}$  are the volume forms of  $Q_k$  and  $L_k$ , respectively, and  $N : Q_k \cup L_k \rightarrow \mathbb{R}^3$  is the normal vector field of  $Q_k \cup L_k$ . Since  $\mathcal{V}_n(f_1) = 0$  and  $Q_k$ , for each  $k \in \mathbb{N}$ , are parts of level sets of  $f_1$ , it follows that  $\mathcal{V}_n$  is tangent to  $Q_k$  for each  $k \in \mathbb{N}$ . This means that the sum above has just the second term, which is clearly bounded by

$$M = \int_L |\langle f_n \mathcal{V}_n, N \rangle| w_L,$$

where  $L$  is the bounded hypersurface given by Definition 4.1, and  $w_L$  is its volume form. From Lemma 4.3 it follows that  $\int_{P_k} \det DF \leq M$ . Then the claim is proved, finishing the proof of the theorem.  $\square$

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